intuitR: A Theorem Prover for Intuitionistic Propositional Logic

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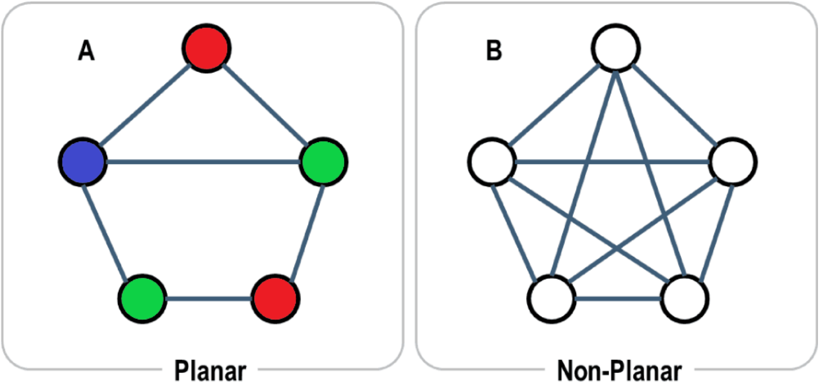
Motivation

Want to write program that assigns each vertex of a planar graph a color, such that no two adjacent vertices are the same color, using smallest number of different colors.



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No more than 4 colors are needed to color a planar graph in this way.

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Proof gives step by step details of how to a color any graph in this way, can write our algorithm based on the proof.
Call this a *constructive proof*.

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$\Downarrow$

Either $\sqrt{2}^{\sqrt{2}}$ or $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ is rational.

But don't know which.

Intuitionistic Logic

Logic system that attempts to emulate/force constructive proofs

Same as classical logic, except it doesn't allow:

- Law of Excluded Middle: $p \vee \neg p$
- Double Negation Elimination: $\neg\neg p \equiv p$

Intuitionistic Propositional Logic (IPL)

Propositional logic form of intuitionistic logic, i.e. no quantifiers ($\forall$ and $\exists$)

Only $\wedge, \vee, \neg, \rightarrow, \bot, \top$

Difficult to determine IPL-validity, so use IPL-provers like *intuitR* by Fiorentini (2021) or *intuit* by Claessen and Rosén (2015)

Outline

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Kripke Models

$(W, \delta, \leq, r)$

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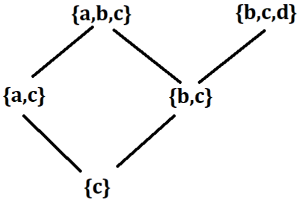

Figure: Visual Representation of a Kripke Model

Forcing

For $k \in W$ and logical formula α, $k \vDash \alpha$ based on the following rules (from Moschovakis (2021)):

1. $k \vDash p, \text{ for every } p \in \delta(k)$
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4. $k \vDash P \vee Q, \text{ if } k \vDash P \text{ or } k \vDash Q$

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4. $k \vDash P \vee Q, \text{ if } k \vDash P \text{ or } k \vDash Q$
5. $k \vDash \neg P, \text{ if for all } k' \geq k, k' \nvDash P$
6. $k \vDash P \rightarrow Q, \text{ if for every } k' \geq k, \text{ if } k' \vDash P, \text{ then } k' \vDash Q$

Determining IPL-validity

A formula α is IPL-valid iff for every kripke model with root r, $r \vDash \alpha$.

Call a kripke model where the root $r \nvDash \alpha$ a *countermodel* for α

Example

Countermodel for $p \vee \neg p$:

$(\{k, k'\}, \delta, \leq, k)$, where

- $\delta(k) = \emptyset$
- $\delta(k') = \{p\}$
- $k < k'$

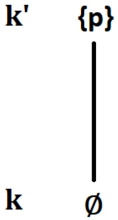

Figure: Visual representation of Kripke Countermodel for $p \vee \neg p$

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When in form $\alpha \rightarrow p$, can reinterpret as "Does α being true make p true?" or "Does α prove p classically?"

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  - Returns:
    - *YES(A')*, if $R(s)$ and $A' \subseteq A$ being true makes g true
    - *NO(M)*, if $R(s)$ and the set of propositional variables $M \supseteq A$ are both true, but g is false

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- X a set of implication clauses: $(a \rightarrow b) \rightarrow c$
- for each $(a \rightarrow b) \rightarrow c \in X$, $b \rightarrow c \in R$
- g a propositional variable

Example

For $p \vee \neg p$:

- $R = \{p \to g, \bot \to g\}$
- $X = \{(p \to \bot) \to g\}$
- $g = g$ (introduced during clausification)

Or $(p \to g) \wedge (\bot \to g) \wedge ((p \to \bot) \to g) \to g$

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cpl_0 and cpl_1

$$\frac{R \vdash_{c} g}{R, X \Rightarrow g}\,\mathrm{cpl}_0$$

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$$\frac{R \vdash_c g}{R, X \Rightarrow g}\ \mathrm{cpl}_0$$

$$\frac{R,\ A \vdash_c b \qquad R,\ \varphi,\ X \Rightarrow g}{R,\ X \Rightarrow g}\ \mathrm{cpl}_1 \qquad \begin{array}{l} (a \to b) \to c \in X \\ A \subseteq V \\ \varphi\ =\ \bigwedge(A \setminus \{a\}) \to c \end{array}$$

Outline

proveR

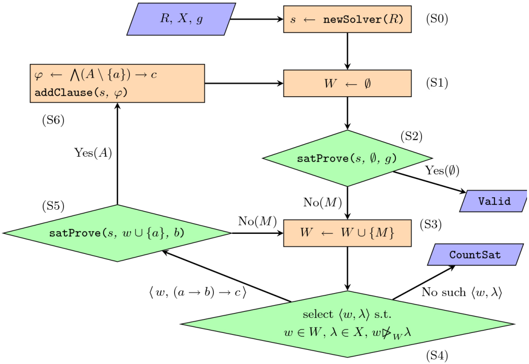
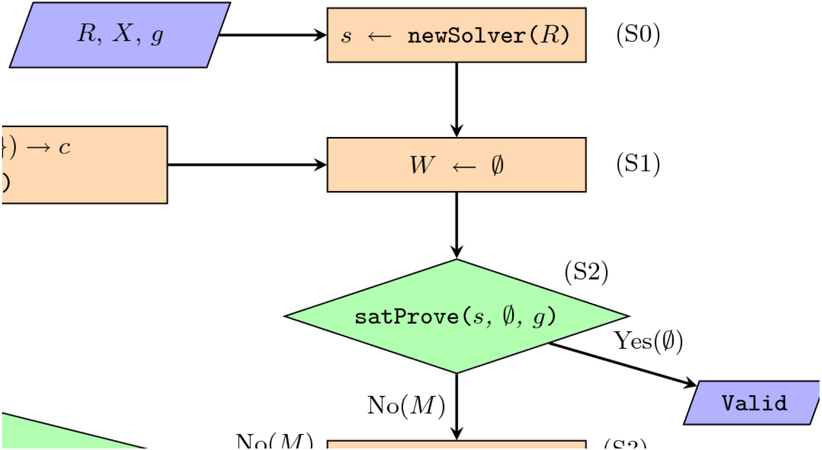
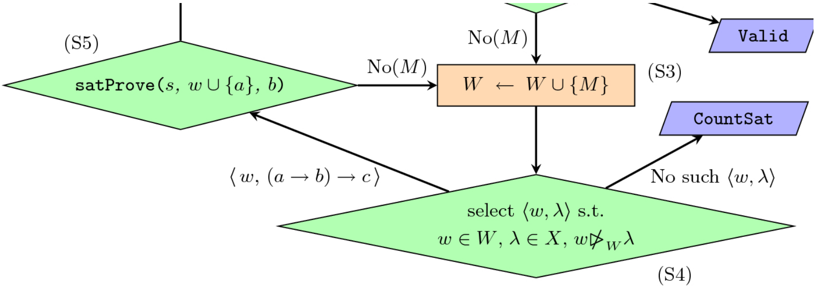


Figure: Flowchart of the *proveR* algorithm

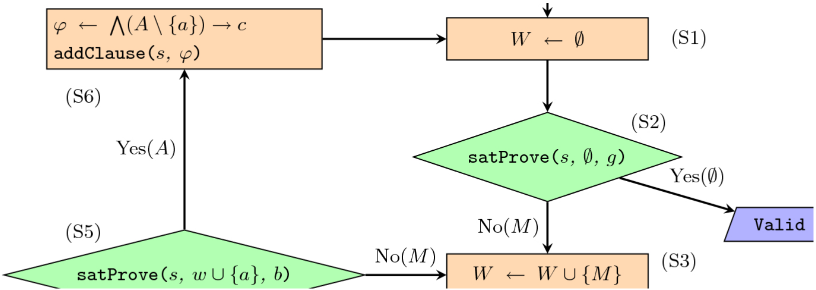
Initialization



Inner Loop


Valid
(S5)
No(M)
No(M)
satProve(s, $w \cup \{a\}$, b)
$W \leftarrow W \cup \{M\}$
(S3)
CountSat
$\langle w, (a \to b) \to c \rangle$
No such $\langle w, \lambda \rangle$
select $\langle w, \lambda \rangle$ s.t.
$w \in W, \lambda \in X, w \not\triangleright_W \lambda$
(S4)

Restart Outer Loop



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Experiment

intuitR was compared to three other IPL-provers:
intuit, *fCube*, and *intHistGC*

Ran on a benchmark set of 1200 problems, split into 32 groups
498 problems were IPL-valid, 702 were not

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Significant Results

Problem Set (Number of Problems)	intuitR	intuit	fCube	intHistGC
SYJ201 (50)	**50 (2.259)**	50 (11.494)	50 (259.776)	50 (39.466)
SYJ207 (50)	**50 (2.291)**	50 (109.919)	50 (138.546)	50 (1014.476)
SYJ211 (50)	**50 (0.462)**	50 (1.251)	50 (1.073)	50 (63.686)
SYJ212 (50)	**50 (0.669)**	42 (587.794)	50 (2.698)	50 (1.624)
EC (100)	100 (2.738)	100 (0.821)	100 (6.183)	**100 (0.651)**
negEC (100)	100 (3.614)	**100 (1.116)**	100 (13.733)	100 (5.807)
portia (100)	100 (32.878)	**100 (22.596)**	100 (3255.818)	100 (3200.135)
Total Unsolved	**28**	36	43	38
Total Time (For These Problems)	**44.911**	734.991	3677.827	4325.845

Table: Most significant results from Fiorentini (2021). Number of problems solved, followed by the time taken to solve said problems (in seconds). Fastest prover highlighted.

Conclusion

Questions?

References

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